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Some properties of the nonlinear matrix equation $X^s + A^*X^{-t}A = Q$ [☆]

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ABSTRACT

In this paper, the nonlinear matrix equation $X^s + A^*X^{-t}A = Q$ is investigated, where Q is a Hermitian positive definite matrix. We consider three cases of this equation: the case $s, t > 0$, the case $s \geqslant 1$, $0 < t \leqslant 1$ and the case $0 < s \leqslant 1$, $t \geqslant 1$. In the case $s, t > 0$, we derive necessary conditions and sufficient condition for the existence of Hermitian positive definite solutions for the matrix equation and obtain some properties of the solutions. As compared to earlier works on these topics, the results we present here are more general, and the analysis here is much simpler. In the cases $s \geqslant 1$, $0 < t \leqslant 1$ and $0 < s \leqslant 1$, $t \geqslant 1$, necessary conditions for the existence of a Hermitian positive definite solution is given, which is sharper than that of Cai and Chen (2010) [9].

1. Introduction

Consider the nonlinear matrix equation

$$X^s + A^*X^{-t}A = Q, \tag{1.1}$$

where A, Q are $n \times n$ complex matrices with Q Hermitian positive definite. Here A^* stands for the conjugate transpose of a matrix A. In this paper, we consider the Hermitian positive definite solutions of this matrix equation with three cases: $s, t > 0$; $s \geqslant 1$, $0 < t \leqslant 1$ and $0 < s \leqslant 1$, $t \geqslant 1$.

Nonlinear matrix equations of the form (1.1) have many applications and have been investigated in some special cases. The case that $s = 1$, $t = 1$ often arises in: control theory; dynamic programming; ladder networks; stochastic filtering; statistics and etc., see [1–3,17] and the references therein.

El-Sayed and her coauthors [11–13] study the case that $s = 1$, $t = \delta$ ($\delta \in (0, 1]$), $s = 1$, $t = n$ and $s = t = 1$. They investigated some properties and the spectral radius of A for the existence of Hermitian positive definite solutions of Eq. (1.1). Researches were made by Liu and Gao [4] for the equations $X^s + A^TX^{-t}A = I$ with $s, t \in N^+$, $A \in R^{n\times n}$ and by Du and Hou [5] for the operator equation $X^m + A^*X^{-n}A = I$ with $m, n \in N^+$, where N^+ is the set of positive integers. The more general case $X^s + A^*X^{-t}A = Q$ ($s, t \in N^+$) has been analyzed in [6–9]. Recently, Duan and Liao [7] discussed the existence of Hermitian positive definite solutions of Eq. (1.1) systematically when A is nonsingular. They investigated the spectral radius of A for the existence of Hermitian positive definite solutions of Eq. (1.1) when $AA^* = A^*A$ and $AQ = QA$. Cai and Chen [8] established and proved theorems for the necessary conditions and some properties for the existence of a Hermitian positive definite solutions of Eq. (1.1) when $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$. They presented a necessary condition on the spectral radius of A for

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the existence of a solution when $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$ and $AA^* = A^*A$. Cai and Chen also investigated the existence of Hermitian positive definite solution of Eq. (1.1) with two cases $s \geqslant 1$, $0 < t \leqslant 1$ and $0 < s \leqslant 1$, $t \geqslant 1$ in [9].

The paper is organized as follows. In Section 2, we discuss the case $s, t > 0$. We present some properties of the Hermitian positive definite solutions, which are more general cases than the one $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$, $s, t \in N^+$ [8] and $Q = I$, $s = 1$, $t = 2$ [10]. We also get a property of the spectral radius of A for the existence of a solution, which is more general case than the one $AQ = QA$, $s, t \in N^+$ [7]; $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$, $s, t \in N^+$ [8] and $Q = I$, $s = 1$, $t = \delta$ [11]. The spectral norm of A for the existence of a Hermitian positive definite solution is given, which generalizes the result of Zhan [3]. As compared to earlier works on these topics, the results we present here are more general, and the analysis here is much simpler. In Section 3, we consider the Hermitian positive definite solutions of Eq. (1.1) with two cases: $0 < t \leqslant 1$ and $0 < s \leqslant 1$, $t \geqslant 1$. Conclusions will be put in Section 4.

The following notations are used throughout the rest of the paper. The notation $B \geqslant 0$ $(B > 0)$ means that B is positive semi-definite (definite) matrix. Moreover, $B \geqslant C$ $(B > C)$ is used as a different notation for $B - C \geqslant 0$ $(B - C > 0)$. This induces a partial ordering on the Hermitian matrices. $X \in [B, C]$ implies that $B \leqslant X \leqslant C$. Let $\lambda_{\max}(H), \lambda_{\min}(H)$ denote the maximal and the minimal eigenvalues of a Hermitian matrix H, respectively. $\lambda(A)$ denotes the eigenvalues of the matrix A. Let $\rho(A)$ stand for the spectral radius of a square matrix A. The norm used in this paper is the spectral norm of the matrix A, i.e., $\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)}$, unless otherwise noted.

Let M be an $n \times n$ real Hermitian positive definite matrix. Then there exists a unique Hermitian positive definite matrix W satisfying $W^t = M$. We denote W as $M^{\frac{1}{t}}$.

For convenience of discussion in the later context, in the sequel, a solution always means a Hermitian positive definite solution.

2. The case $s, t > 0$

In this section, we consider the case $s, t > 0$.

Lemma 2.1. *(See [8, Theorem 2.2].) Eq.* (1.1) $(s, t \in N^+)$ *has a solution if and only if there exist unitary matrices U, P and diagonal matrices $C = \operatorname{diag}(c_1, c_2, \ldots, c_n) > 0$ and $S = \operatorname{diag}(s_1, s_2, \ldots, s_n) \geqslant 0$ with $C^2 + S^2 = I$ such that*

$$A = \left(Q^{\frac{1}{2}}P^*C^2PQ^{\frac{1}{2}}\right)^{\frac{t}{2s}}USPQ^{\frac{1}{2}}.$$

In this case $X = (Q^{\frac{1}{2}}P^*C^2PQ^{\frac{1}{2}})^{\frac{1}{s}}$ is a solution.

The extension of Lemma 2.1 to the case $s, t > 0$ is straightforward.

Theorem 2.1. *If A is singular and Eq.* (1.1) *has a solution X, then*

$$\lambda_{\max}\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = 1.$$

Proof. *Method* 1. Eq. (1.1) is equivalent to

$$Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}} + Q^{-\frac{1}{2}}A^*X^{-t}AQ^{-\frac{1}{2}} = I. \tag{2.1}$$

If A is singular, then $AQ^{-\frac{1}{2}}$ is singular. According to linear algebraic theorem, there exists a vector $y \in C^n$, $y \neq 0$, such that $AQ^{-\frac{1}{2}}y = 0$. Multiplying right side of (2.1) by y, we have

$$Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}y + Q^{-\frac{1}{2}}A^*X^{-t}AQ^{-\frac{1}{2}}y = y.$$

It is equivalent to

$$Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}y = y.$$

Then we have $1 \in \lambda(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}})$.

On the other hand, by (2.1), one can easily see that $\lambda_{\max}(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}) \leqslant 1$. Consequently, we have $\lambda_{\max}(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}) = 1$.

Method 2. According to Lemma 2.1, we have $A = (Q^{\frac{1}{2}}P^*C^2PQ^{\frac{1}{2}})^{\frac{t}{2s}}USPQ^{\frac{1}{2}}$. Since A is singular, we know that S is singular. There exist some diagonal entries $s_i = 0$. Since $C^2 + S^2 = I$, there exist some $c_i = 1$. From $X = (Q^{\frac{1}{2}}P^*C^2PQ^{\frac{1}{2}})^{\frac{1}{s}}$, we have $Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}} = P^*C^2P$. Then

$$\lambda\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right) = \lambda\left(P^*C^2P\right) = \lambda\left(PP^*C^2\right) = \lambda\left(C^2\right).$$

And we get

$$1 \in \lambda\left(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}\right).$$

On the other hand, by (2.1), one can easily see that $\lambda_{\max}(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}) \leqslant 1$. Consequently, we have $\lambda_{\max}(Q^{-\frac{1}{2}}X^sQ^{-\frac{1}{2}}) = 1$. □

Remark 2.1. Since Theorem 3.2 in [8] has the conditions $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$ and $s, t \in N^+$, Theorem 2.1 generalizes the existing related result. Also, the analysis here is much simpler.

Remark 2.2. Let $s = 1, t = 2$ and $Q = I$ in Theorem 2.1, we get Lemma 3 in [10].

Corollary 2.2. *If A is singular and Eq. (1.1) has a solution X, the following inequality holds*:

$$\lambda_{\min}^{\frac{1}{s}}(Q) \leqslant \lambda_{\max}(X) \leqslant \lambda_{\max}^{\frac{1}{s}}(Q). \tag{2.2}$$

Proof. The proof is similar to that of Corollary 3.3 of Cai and Chen [8] and is omitted here. □

Remark 2.3. Corollary 3.3 of Cai and Chen in [8] with the case $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$ and $s, t \in N^+$ is a special case of Corollary 2.2.

Lemma 2.2. *(See [7, Lemma 3.1].) Let $f(x) = x^t(\eta - x^s)$, $\eta > 0$, $x \geqslant 0$. Then*

(i) *f is increasing on $[0, (\frac{t}{s+t}\eta)^{\frac{1}{s}}]$ and decreasing on $[(\frac{t}{s+t}\eta)^{\frac{1}{s}}, +\infty]$.*
(ii) $f_{\max} = f((\frac{t}{s+t}\eta)^{\frac{1}{s}}) = \frac{s}{s+t}(\frac{t}{s+t})^{\frac{t}{s}}\eta^{\frac{t}{s}+1}$.

Theorem 2.3. *Assume that Eq. (1.1) is solvable. Then the matrix A satisfies the following inequality*

$$\big(\rho(A)\big)^2 \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_{\max}^{\frac{t}{s}+1}(Q).$$

Proof. For any eigenvalue λ of A, let y be an eigenvector corresponding to λ with $\|y\|_2 = 1$. Multiplying left side of Eq. (1.1) by y^* and right side by y, we have

$$y^*X^sy + y^*A^*X^{-t}Ay = y^*Qy,$$

which yields

$$y^*X^sy + |\lambda|^2y^*X^{-t}y = y^*Qy. \tag{2.3}$$

Next, the proof of this theorem follows the same lines as the proof of Theorem 3.1 in [8]. Since X is a Hermitian positive definite matrix, there exists a unitary matrix U such that $X = U\Lambda U^*$, where $\Lambda = \operatorname{diag}(\eta_1, \dots, \eta_n) > 0$. Then (2.3) turns into the following form:

$$y^*U\Lambda^sU^*y + |\lambda|^2y^*U\Lambda^{-t}U^*y = y^*Qy. \tag{2.4}$$

Now, introduce the variable $\nu = (\nu_1, \nu_2, \dots, \nu_n)^T = U^*y$. Then (2.4) reduces to

$$\nu^*\Lambda^s\nu + |\lambda|^2\nu^*\Lambda^{-t}\nu = (U\nu)^*QU\nu,$$

from which we obtain

$$|\lambda|^2 = \frac{\nu^*(U^*QU - \Lambda^s)\nu}{\nu^*\Lambda^{-t}\nu} \leqslant \frac{\nu^*(\lambda_{\max}(Q)I - \Lambda^s)\nu}{\nu^*\Lambda^{-t}\nu} = \frac{\sum_{i=1}^n |\nu_i|^2(\lambda_{\max}(Q) - \eta_i^s)}{\sum_{i=1}^n |\nu_i|^2\eta_i^{-t}}.$$

According to Lemma 2.2, we know that

$$\eta_i^t\big(\lambda_{\max}(Q) - \eta_i^s\big) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_{\max}^{\frac{t}{s}+1}(Q),$$

i.e.,

$$\lambda_{\max}(Q) - \eta_i^s \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_{\max}^{\frac{t}{s}+1}(Q)\eta_i^{-t}.$$

Noting that $\nu \neq 0$, we get

$$\sum_{i=1}^n |\nu_i|^2\big(\lambda_{\max}(Q) - \eta_i^s\big) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_{\max}^{\frac{t}{s}+1}(Q)\sum_{i=1}^n |\nu_i|^2\eta_i^{-t}.$$

Consequently,

$$|\lambda|^2 = \frac{\sum_{i=1}^{n} |\nu_i|^2(\lambda_{\max}(Q) - \eta_i^s)}{\sum_{i=1}^{n} |\nu_i|^2\eta_i^{-t}} \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_{\max}^{\frac{t}{s}+1}(Q).$$

Then

$$\bigl(\rho(A)\bigr)^2 \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_{\max}^{\frac{t}{s}+1}(Q). \qquad \square$$

Remark 2.4. Theorem 2.3 generalizes Theorem 3.1 of Cai and Chen [8] which has the condition $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$ and $s,t \in N^+$. If we take $s=1$, $t=\delta$, $Q=I$, we get

$$\rho(A)^2 \leqslant \frac{\delta^\delta}{(\delta+1)^{(\delta+1)}}.$$

This is Theorem 2.2 of Hasanov and El-Sayed [11]. It is a special case of Theorem 2.3.

Corollary 2.4. *Let $AA^* = A^*A$. If Eq.* (1.1) *has a solution X, then*

$$\lambda_{\max}\bigl(A^*A\bigr) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_{\max}^{\frac{t}{s}+1}(Q).$$

Proof. Since $AA^* = A^*A$, then $\lambda_{\max}(A^*A) = (\rho(A))^2$. According to Theorem 2.3, we conclude that

$$\lambda_{\max}\bigl(A^*A\bigr) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_{\max}^{\frac{t}{s}+1}(Q). \qquad \square$$

Remark 2.5. Theorem 4.1 of Duan and Liao [7] with the case $AQ = QA$ and $s,t \in N^+$ is a special case of Corollary 2.4. If we take $s=t=1$, $Q=I$, we get Theorem 11 of Engwerda [1]. If we take $s=1$, $t=2$, $Q=I$, we get Theorem 1 of Zhang [10]. Corollary 2.4 generalizes the existing related result.

Theorem 2.5. *Assume Eq.* (1.1) *has a solution X. Let $\lambda(A)$ be any eigenvalue of A.*

(i) *If A is nonsingular, then*

$$\lambda_{\min}^t(X)\bigl(\lambda_{\min}(Q) - \lambda_{\max}^s(X)\bigr) \leqslant \bigl|\lambda(A)\bigr|^2 \leqslant \lambda_{\max}^t(X)\bigl(\lambda_{\max}(Q) - \lambda_{\min}^s(X)\bigr); \tag{2.5}$$

(ii) *If A is singular, then*

$$0 \leqslant \bigl|\lambda(A)\bigr|^2 \leqslant \lambda_{\max}^t(X)\bigl(\lambda_{\max}(Q) - \lambda_{\min}^s(X)\bigr). \tag{2.6}$$

Proof. (i) Let y be an eigenvector corresponding to $\lambda(A)$ with $\|y\|_2 = 1$. Then by (2.3), we have

$$y^*X^sy + \bigl|\lambda(A)\bigr|^2 y^*X^{-t}y = y^*Qy,$$

which yields that

$$\bigl|\lambda(A)\bigr|^2 = \frac{y^*Qy - y^*X^sy}{y^*X^{-t}y}.$$

Then we have

$$\min_{y\in C^n,\ \|y\|_2=1}\left(\frac{y^*Qy - y^*X^sy}{y^*X^{-t}y}\right) \leqslant \bigl|\lambda(A)\bigr|^2 \leqslant \max_{y\in C^n,\ \|y\|_2=1}\left(\frac{y^*Qy - y^*X^sy}{y^*X^{-t}y}\right).$$

Since

$$\begin{aligned}
\min_{y\in C^n,\ \|y\|_2=1}\left(\frac{y^*Qy - y^*X^sy}{y^*X^{-t}y}\right) &\geqslant \frac{\min_{y\in C^n,\ \|y\|_2=1}(y^*Qy - y^*X^sy)}{\max_{y\in C^n,\ \|y\|_2=1}(y^*X^{-t}y)} \\
&\geqslant \frac{\min_{y\in C^n,\ \|y\|_2=1}(y^*Qy) - \max_{y\in C^n,\ \|y\|_2=1}(y^*X^sy)}{\max_{y\in C^n,\ \|y\|_2=1}(y^*X^{-t}y)} \\
&= \lambda_{\min}^t(X)\bigl(\lambda_{\min}(Q) - \lambda_{\max}^s(X)\bigr);
\end{aligned}$$

$$\begin{aligned}\max_{y\in C^n,\,\|y\|_2=1}\left(\frac{y^*Qy-y^*X^sy}{y^*X^{-t}y}\right) &\leqslant \frac{\max_{y\in C^n,\,\|y\|_2=1}(y^*Qy-y^*X^sy)}{\min_{y\in C^n,\,\|y\|_2=1}(y^*X^{-t}y)}\\ &\leqslant \frac{\max_{y\in C^n,\,\|y\|_2=1}(y^*Qy)-\min_{y\in C^n,\,\|y\|_2=1}(y^*X^sy)}{\min_{y\in C^n,\,\|y\|_2=1}(y^*X^{-t}y)}\\ &=\lambda_{\max}^t(X)\big(\lambda_{\max}(Q)-\lambda_{\min}^s(X)\big).\end{aligned}$$

Then we have

$$\lambda_{\min}^t(X)\big(\lambda_{\min}(Q)-\lambda_{\max}^s(X)\big)\leqslant\big|\lambda(A)\big|^2\leqslant\lambda_{\max}^t(X)\big(\lambda_{\max}(Q)-\lambda_{\min}^s(X)\big).$$

(ii) Since A is singular, by Corollary 2.2, we see that left side of the inequality (2.5) always holds. Then we get (2.6). $\square$

Remark 2.6. Theorem 3.3 of Cai and Chen [8] with the case $AQ^{\frac{1}{2}}=Q^{\frac{1}{2}}A$ and $s,t\in N^+$, Theorem 1 of S.M. El-Sayed and A.M. Al-Dbiban [12] with the case $s=1$, $t=n$, and Theorem 2.4 of S.M. El-Sayed [13] with the case $s=t=1$, $Q=I$, are special cases of Theorem 2.5.

Since $Q>0$, there exist a unitary matrix V such that $Q=V\Sigma V^*$, where $\Sigma=\mathrm{diag}(\mu_1,\mu_2,\dots,\mu_n)$, $\mu_1\geqslant\mu_2\geqslant\cdots\geqslant\mu_n>0$.

Theorem 2.6. *If Eq.* (1.1) *has a solution, then*

$$\|A\|_2<(\mu_1)^{\frac{s+t}{2s}}.$$

Moreover, if A has order $\geqslant 2$, then $\|A\|_2$ can take any value in the interval $[0,b)$ for Eq. (1.1) *to have a solution, where $b=\max\{\mu_1^{\frac{t}{2s}}\mu_2^{\frac{1}{2}},\mu_1^{\frac{1}{2}}\mu_2^{\frac{t}{2s}}\}$.*

Proof. According to Lemma 2.1, we have

$$A=\big(Q^{\frac{1}{2}}P^*C^2PQ^{\frac{1}{2}}\big)^{\frac{t}{2s}}USPQ^{\frac{1}{2}}.$$

Taking norm of each side, we get

$$\begin{aligned}\|A\|_2&=\big\|\big(Q^{\frac{1}{2}}P^*C^2PQ^{\frac{1}{2}}\big)^{\frac{t}{2s}}USPQ^{\frac{1}{2}}\big\|_2\\ &\leqslant\big\|Q^{\frac{1}{2}}P^*C^2PQ^{\frac{1}{2}}\big\|_2^{\frac{t}{2s}}\|U\|_2\|S\|_2\|P\|_2\big\|Q^{\frac{1}{2}}\big\|_2\\ &\leqslant\big(\|C\|_2^2\big\|Q^{\frac{1}{2}}\big\|_2^2\big)^{\frac{t}{2s}}\|S\|_2\big\|Q^{\frac{1}{2}}\big\|_2\\ &\leqslant\|C\|_2^{\frac{t}{s}}\|Q\|_2^{\frac{s+t}{2s}}\|S\|_2\\ &<\|Q\|_2^{\frac{s+t}{2s}}=(\mu_1)^{\frac{s+t}{2s}}.\end{aligned}$$

The last inequality follows from $\|C\|_2\leqslant 1$ and $\|S\|_2<1$, which is easily seen from the fact that $C^2+S^2=I$ and $C>0$.

To prove the second statement we consider the 2×2 matrix

$$\begin{aligned}A_{21}&=V\begin{pmatrix}0 & a\mu_1^{\frac{t}{2s}}\mu_2^{\frac{1}{2}}\\ 0 & 0\end{pmatrix}V^*\\ &=\left(Q^{\frac{1}{2}}V\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-a^2}\end{pmatrix}^2V^*Q^{\frac{1}{2}}\right)^{\frac{t}{2s}}V\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & a\end{pmatrix}V^*Q^{\frac{1}{2}},\end{aligned}$$

or

$$\begin{aligned}A_{22}&=V\begin{pmatrix}0 & 0\\ a\mu_1^{\frac{1}{2}}\mu_2^{\frac{t}{2s}} & 0\end{pmatrix}V^*\\ &=\left(Q^{\frac{1}{2}}V\begin{pmatrix}\sqrt{1-a^2} & 0\\ 0 & 1\end{pmatrix}^2V^*Q^{\frac{1}{2}}\right)^{\frac{t}{2s}}V\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}V^*Q^{\frac{1}{2}},\end{aligned}$$

where $0\leqslant a<1$. Then $\|A_{21}\|_2=a\mu_1^{\frac{t}{2s}}\mu_2^{\frac{1}{2}}$ or $\|A_{22}\|_2=a\mu_1^{\frac{1}{2}}\mu_2^{\frac{t}{2s}}$. According to Lemma 2.1, Eq. (1.1) always has a solution for $A=A_{21}$ or $A=A_{22}$ with any $\|A\|_2\in[0,b)$, where $b=\max\{\mu_1^{\frac{t}{2s}}\mu_2^{\frac{1}{2}},\mu_1^{\frac{1}{2}}\mu_2^{\frac{t}{2s}}\}$. For higher matrix orders n just use $A=V(B_2\oplus 0_{n-2})V^*$, where $B_2=\begin{pmatrix}0 & 0\\ a\mu_1^{\frac{1}{2}}\mu_2^{\frac{t}{2s}} & 0\end{pmatrix}$ or $B_2=\begin{pmatrix}0 & a\mu_1^{\frac{t}{2s}}\mu_2^{\frac{1}{2}}\\ 0 & 0\end{pmatrix}$. $\square$

Remark 2.7. If $\mu_1 = \mu_2 = \lambda_{\max}(Q)$, then $\|A_2\|_2 = a\lambda_{\max}^{\frac{s+t}{2s}}(Q)$, where $a \in [0,1)$. That is to say the norm of A can take any value in the interval $[0, \lambda_{\max}^{\frac{s+t}{2s}}(Q))$. Theorem 3.1 of Zhan [3] is a special case of Theorem 2.6. Thus the existing related results are deepened and extended.

3. Cases $s \geqslant 1$, $0 < t \leqslant 1$ and $0 < s \leqslant 1$, $t \geqslant 1$

In this section, we shall derive necessary condition for the Hermitian positive definite solutions of Eq. (1.1) with two cases: $s \geqslant 1$, $0 < t \leqslant 1$ and $0 < s \leqslant 1$, $t \geqslant 1$.

Lemma 3.1. *(See [9, Theorem 2.1].) Let X be a Hermitian positive definite solution of Eq.* (1.1)*. A is nonsingular.*

(1) *If $s \geqslant 1, 0 < t \leqslant 1$, then $(\frac{\lambda_{\min}(AQ^{-1}A^*)}{\lambda_{\max}(AQ^{-1}A^*)})^{\frac{1-t}{t}}(AQ^{-1}A^*)^{\frac{1}{t}} < X < (Q - A^*Q^{-\frac{t}{s}}A)^{\frac{1}{s}}$;*
(2) *If $0 < s \leqslant 1$ and $t \geqslant 1$, then $(AQ^{-1}A^*)^{\frac{1}{t}} < X < (\frac{\mu}{\nu})^{\frac{1-s}{t}}(Q - (\frac{\lambda_{\min}Q}{\lambda_{\max}Q})^{\frac{t}{s}-1}A^*Q^{-\frac{t}{s}}A)^{\frac{1}{s}})$, where μ and ν are the maximal and the minimal eigenvalues of the matrix $Q - (\frac{\lambda_{\min}Q}{\lambda_{\max}Q})^{\frac{t}{s}-1}A^*Q^{-\frac{t}{s}}A$, respectively.*

Lemma 3.2. *(See [14, Theorem 1.1, Lowner–Heinz].) If $B > C > 0$ (or $B \geqslant C > 0$), then $B^\alpha > C^\alpha > 0$ (or $B^\alpha \geqslant C^\alpha > 0$) for all $\alpha \in (0,1]$, and $C^\alpha > B^\alpha > 0$ (or $C^\alpha \geqslant B^\alpha > 0$) for all $\alpha \in [-1, 0)$.*

Lemma 3.3. *(See [15, p. 656].) For square nonsingular matrices A, B and C, application of Schur's lemma to the two matrices $A + BC$ and $A - BC$ yields that*

(i) $(A + BC)^{-1} = A^{-1} - A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1}$;
(ii) $(A - BC)^{-1} = A^{-1} - A^{-1}B(CA^{-1}B - I)^{-1}CA^{-1}$.

Lemma 3.4. *(See [16, Theorem 2.1].) Let A and B be positive operators on a Hilbert space H such that $M_1 I \geqslant A \geqslant m_1 I > 0$, $M_2 I \geqslant B \geqslant m_2 I > 0$ and $B \geqslant A > O$. Then*

$$A^t \leqslant \left(\frac{M_1}{m_1}\right)^{t-1} B^t, \qquad A^t \leqslant \left(\frac{M_2}{m_2}\right)^{t-1} B^t$$

hold for any $t \geqslant 1$.

Now we give new estimates which are sharper than Lemma 2.1.

Theorem 3.1. *Let X be a Hermitian positive definite solution of Eq.* (1.1)*. A is nonsingular.*

(1) *If $s \geqslant 1$, $0 < t \leqslant 1$, then*

$$\left(\frac{\lambda_{\min}(B)}{\lambda_{\max}(B)}\right)^{\frac{1-t}{t}} (B)^{\frac{1}{t}} < X < \left(Q - A^*Q^{-\frac{t}{s}}A\right)^{\frac{1}{s}},$$

where

$$B = AQ^{-1}A^* + AQ^{-1}\left(\eta\left(AQ^{-1}A^*\right)^{-\frac{s}{t}} - Q^{-1}\right)^{-1}Q^{-1}A^*$$

and

$$\eta = \left(\frac{\lambda_{\max}(AQ^{-1}A^*)}{\lambda_{\min}(AQ^{-1}A^*)}\right)^{\frac{s-t}{t}}.$$

(2) *$0 < s \leqslant 1$ and $t \geqslant 1$, then*

$$\left(AQ^{-1}A^* + AQ^{-1}\left(\left(AQ^{-1}A^*\right)^{-\frac{s}{t}} - Q^{-1}\right)^{-1}Q^{-1}A^*\right)^{\frac{1}{t}} < X < \left(\frac{\mu}{\nu}\right)^{\frac{1-s}{s}} \left(Q - \left(\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}\right)^{\frac{t}{s}-1} A^*Q^{-\frac{t}{s}}A\right)^{\frac{1}{s}},$$

where μ and ν are the maximal and the minimal eigenvalues of the matrix $Q - (\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)})^{\frac{t}{s}-1}A^*Q^{-\frac{t}{s}}A$, respectively.

Proof. (1) The proof in (1) of the upper bound can be found from Theorem 2.1 in Cai and Chen [9]. It is omitted. Assume X is a Hermitian positive definite solution of Eq. (1.1). If $s \geqslant 1$ and $0 < t \leqslant 1$, then it follows from $0 < X^s < Q$ and Lemma 3.2 that

$$Q^{-1} < X^{-s}. \tag{3.1}$$

On the other hand, from $A^*X^{-t}A < Q$, it follows that

$$Q^{-\frac{1}{2}}A^*X^{-\frac{t}{2}}X^{-\frac{t}{2}}AQ^{-\frac{1}{2}} < I,$$
$$X^{-\frac{t}{2}}AQ^{-1}A^*X^{-\frac{t}{2}} < I,$$
$$AQ^{-1}A^* < X^t. \tag{3.2}$$

Since $\lambda_{\min}(AQ^{-1}A^*)I \leqslant AQ^{-1}A^* \leqslant \lambda_{\max}(AQ^{-1}A^*)I$ and $\frac{s}{t} \geqslant 1$, applying Lemma 3.4 to (3.2) yields

$$\left(AQ^{-1}A^*\right)^{\frac{s}{t}} < \left(\frac{\lambda_{\max}(AQ^{-1}A^*)}{\lambda_{\min}(AQ^{-1}A^*)}\right)^{\frac{s-t}{t}} X^s.$$

Let $\eta = (\frac{\lambda_{\max}(AQ^{-1}A^*)}{\lambda_{\min}(AQ^{-1}A^*)})^{\frac{s-t}{t}}$, by Lemma 3.2, we get

$$X^{-s} < \eta\left(AQ^{-1}A^*\right)^{\frac{-s}{t}}. \tag{3.3}$$

Rewriting Eq. (1.1), we have

$$X^t = A\left(Q - X^s\right)^{-1}A^*. \tag{3.4}$$

Applying Lemma 3.3 to (3.4) and combining (3.1) and (3.3) yield

$$\begin{aligned} X^t &= A\left(Q - X^s\right)^{-1}A^* \\ &= A\left(Q^{-1} - Q^{-1}\left(X^sQ^{-1} - I\right)^{-1}X^sQ^{-1}\right)A^* \\ &= AQ^{-1}A^* + AQ^{-1}\left(X^{-s} - Q^{-1}\right)^{-1}Q^{-1}A^* \\ &> AQ^{-1}A^* + AQ^{-1}\left(\eta\left(AQ^{-1}A^*\right)^{-\frac{s}{t}} - Q^{-1}\right)^{-1}Q^{-1}A^*. \end{aligned} \tag{3.5}$$

Let

$$B = AQ^{-1}A^* + AQ^{-1}\left(\eta\left(AQ^{-1}A^*\right)^{-\frac{s}{t}} - Q^{-1}\right)^{-1}Q^{-1}A^*. \tag{3.6}$$

Since $\frac{1}{t} \geqslant 1$ and $\lambda_{\min}(B)I \leqslant B \leqslant \lambda_{\max}(B)I$, combining Lemma 3.4, (3.5) and (2.2) yields

$$(B)^{\frac{1}{t}} < \left(\frac{\lambda_{\max}(B)}{\lambda_{\min}(B)}\right)^{\frac{1-t}{t}} X,$$
$$\left(\frac{\lambda_{\min}(B)}{\lambda_{\max}(B)}\right)^{\frac{1-t}{t}} (B)^{\frac{1}{t}} < X.$$

(2) The proof in (2) of the upper bound can be found from Theorem 2.1 in Cai and Chen [9]. It is omitted. Assume X is a Hermitian positive definite solution of Eq. (1.1). If $0 < s \leqslant 1$ and $t \geqslant 1$, then it follows from $0 < X^s < Q$ and Lemma 3.2 that

$$Q^{-1} < X^{-s}. \tag{3.7}$$

On the other hand, from $A^*X^{-t}A < Q$, it follows that

$$Q^{-\frac{1}{2}}A^*X^{-\frac{t}{2}}X^{-\frac{t}{2}}AQ^{-\frac{1}{2}} < I,$$
$$X^{-\frac{t}{2}}AQ^{-1}A^*X^{-\frac{t}{2}} < I,$$
$$AQ^{-1}A^* < X^t.$$

Since $0 < \frac{s}{t} \leqslant 1$, by Lemma 3.2, we get

$$X^{-s} < \left(AQ^{-1}A^*\right)^{-\frac{s}{t}}. \tag{3.8}$$

Applying Lemma 3.3 to (3.4) and combining (3.7) and (3.8) yield

$$\begin{aligned} X^t &= A\left(Q - X^s\right)^{-1}A^* \\ &= A\left(Q^{-1} - Q^{-1}\left(X^sQ^{-1} - I\right)^{-1}X^sQ^{-1}\right)A^* \\ &= AQ^{-1}A^* + AQ^{-1}\left(X^{-s} - Q^{-1}\right)^{-1}Q^{-1}A^* \\ &> AQ^{-1}A^* + AQ^{-1}\left(\left(AQ^{-1}A^*\right)^{-\frac{s}{t}} - Q^{-1}\right)^{-1}Q^{-1}A^*. \end{aligned} \tag{3.9}$$

Since $0 < \frac{1}{t} \leqslant 1$, combining Lemma 3.2 and (3.9) yields

$$\left(AQ^{-1}A^* + AQ^{-1}\left(\left(AQ^{-1}A^*\right)^{-\frac{s}{t}} - Q^{-1}\right)^{-1}Q^{-1}A^*\right)^{\frac{1}{t}} < X. \qquad \square$$

Remark 3.1. Comparing Theorem 3.4 with Lemma 3.1, it is easy to obtain that

$$\left(AQ^{-1}A^*\right)^{\frac{1}{t}} < \left(AQ^{-1}A^* + AQ^{-1}\left(\left(AQ^{-1}A^*\right)^{-\frac{s}{t}} - Q^{-1}\right)^{-1}Q^{-1}A^*\right)^{\frac{1}{t}},$$

when $0 < s \leqslant 1$ and $t \geqslant 1$. That is to say, our estimate of Hermitian positive definite solution of Eq. (1.1) is sharper than that of Cai and Chen [9]. If $s = 1$, $t \geqslant 1$, we have

$$\left(AQ^{-1}A^* + AQ^{-1}\left(\left(AQ^{-1}A^*\right)^{-\frac{1}{t}} - Q^{-1}\right)^{-1}Q^{-1}A^*\right)^{\frac{1}{t}} < X < Q - \left(\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}\right)^{t-1} A^*Q^{-t}A.$$

This is Theorem 2.3 in [18]. A large number of numerical experiments shows that the following inequality

$$\left(\frac{\lambda_{\min}(AQ^{-1}A^*)}{\lambda_{\max}(AQ^{-1}A^*)}\right)^{\frac{1-t}{t}} \left(AQ^{-1}A^*\right)^{\frac{1}{t}} \leqslant \left(\frac{\lambda_{\min}(B)}{\lambda_{\max}(B)}\right)^{\frac{1-t}{t}} (B)^{\frac{1}{t}}$$

holds when $s \geqslant 1$ and $0 < t \leqslant 1$. However, it is very difficult to prove this inequality, which remains as an unsolved problem.

Corollary 3.2. *If $s = t = 1$, according to Theorem 3.1, we get*

$$AQ^{-1}A^* + AQ^{-1}\left(\left(AQ^{-1}A^*\right)^{-1} - Q^{-1}\right)^{-1}Q^{-1}A^* < X < Q - A^*Q^{-1}A.$$

4. Conclusions

In this paper, we consider Eq. (1.1) with three cases: $s, t > 0$; $s \geqslant 1$, $0 < t \leqslant 1$ and $0 < s \leqslant 1$, $t \geqslant 1$. In the case $s, t > 0$, we present some properties of the Hermitian positive definite solutions, which are more general cases than the one $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$, $s, t \in N^+$ [8] and $Q = I$, $s = 1$, $t = 2$ [10]. We also get a property of the spectral radius of A for the existence of a solution, which is more general case than the one $AQ = QA$, $s, t \in N^+$ [7]; $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$, $s, t \in N^+$ [8] and $Q = I$, $s = 1$, $t = \delta$ [11]. The spectral norm of A for the existence of a Hermitian positive definite solution is given, which generalizes the result of Zhan [3]. As compared to earlier works on these topics, the results we present here are more general, and the analysis here is much simpler. We also derive some necessary conditions for the matrix equation to have a Hermitian positive definite solution with two cases $s \geqslant 1$, $0 < t \leqslant 1$ and $0 < s \leqslant 1$, $t \geqslant 1$, which is sharper than that of Cai and Chen [9].

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